PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Paul T. Bateman, Mario Benedicty, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, Jerrold Grossman, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before March 31, 2009. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11390. *Proposed by Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI.* Let G be the undirected graph on the vertex set V of all pairs (a, b) of relatively prime integers, with edges linking (a, b) to $(a + kab, b)$ and $(a, b + kab)$ for all integers k.
(**a**) Show that for all (a, b) in V, there is a path joining (a, b) and $(1, 1)$.
(**b**)* Call an edge linking (a, b) to $(a + kab, b)$ or $(a, b + kab)$ *positive* if $k > 0$, and *negative* if $k < 0$. Let the reversal number of a path from $(1, 1)$ to (a, b) be one more than the number of sign changes along the path, and let the reversal value of (a, b) be the minimal reversal number over all paths from $(1, 1)$ to (a, b). Are there pairs of arbitrarily high reversal value?

11391. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bîrlad, Romania.* Let p be a positive prime and s a positive integer. Let n and k be integers such that $n \geq k \geq p^s - p^{s-1}$, and let $x_1, \ldots, x_n$ be integers. For $1 \leq j \leq n$, let m_j be the number of expressions of the form $x_{i_1} + \cdots + x_{i_j}$ with $1 \leq i_1 < \cdots < i_j \leq n$ that evaluate to 0 modulo p, and let n_j denote the number of such expressions that do not. (Set $m_0 = 1$ and $n_0 = 0$). Apart from the cases $(s, k) = (1, p - 1)$ and $s = p = k = 2$, show that

$$\sum_{j=0}^{k} (-1)^j \binom{n-k+j}{j} m_{k-j} \equiv 0 \pmod{p^s},$$

and show that the same congruence holds with n_{k-j} in place of m_{k-j}.

11392. *Proposed by Omran Kouba, Higher Institute for Applied Science and Technology, Damascus, Syria.* Let the consecutive vertices of a regular n-gon P be denoted $A_0, \ldots, A_{n-1}$, in order, and let $A_n = A_0$. Let M be a point such that for $0 \leq k < n$ the perpendicular projections of M onto each line $A_k A_{k+1}$ lie interior to the segment (A_k, A_{k+1}). Let B_k be the projection of M onto $A_k A_{k+1}$. Show that

$$\sum_{k=0}^{n-1} \text{Area}(\triangle(M A_k B_k)) = \frac{1}{2}\text{Area}(P).$$

11393. *Proposed by Cosmin Pohoata (student), National College "Tudor Vianu", Bucharest, Romania.* In triangle ABC, let M and Q be points on segment AB, and similarly let N and R be points on AC, and P and S be points on BC. Let d_1 be the line through M and N, d_2 the line through P and Q, and d_3 the line through R and S. Let $\rho(X, Y, Z)$ denote the ratio of the length of XZ to that of XY. Let $m = \rho(M, A, B)$, $n = \rho(N, A, C)$, $p = \rho(P, B, C)$, $q = \rho(Q, B, A)$, $r = \rho(R, C, A)$, and $s = \rho(S, C, B)$. Prove that the lines (d_1, d_2, d_3) are concurrent if and only if $mpr + nqs + mq + nr + ps = 1$.

11394. *Proposed by K. S. Bhanu, Institute of Science, Nagpur, India, and M. N. Deshpande, Nagpur, India.* A fair coin is tossed n times, with $n \geq 2$. Let R be the resulting number of runs of the same face, and X the number of isolated heads. Show that the covariance of the random variables R and X is $n/8$.

11395. *Proposed by M. Farrokhi D. G.,University of Tsukuba, Tsukuba, Japan.* Prove that if H is a finite subgroup of the group G of all continuous bijections of $[0, 1]$ to itself, then the order of H is 1 or 2.

11396. *Proposed by Gérard Letac, Université Paul Sabatier, Toulouse, France.* For complex z, let $H_n(z)$ denote the $n \times n$ Hermitian matrix whose diagonal elements all equal 1 and whose above-diagonal elements all equal z. For $n \geq 2$, find all z such that $H_n(z)$ is positive semi-definite.

SOLUTIONS

Generalizing $(1-1)^n = 0$

11230 [2006, 567]. *Proposed by Gregory Keselman, Oak Park, MI, formerly of Lvov Polytechnic Institute, Ukraine.* Let n be a positive integer, let $[n] = \{0, 1, \ldots, n-1\}$, and for a subset P of $[n]$ let $s(z, P) = \sum_{j \in P} z^j$. With the usual understanding that $0^0 = 1$, show that

$$\sum_{P \subseteq [n]} (-1)^{|P|} s^k(z, P) = 0 \qquad (k < n),$$
$$\sum_{P \subseteq [n]} (-1)^{|P|} s^n(z, P) = (-1)^n (n!) z^{n(n-1)/2},$$
$$\sum_{P \subseteq [n]} (-1)^{|P|} s^{n+1}(z, P) = (-1)^n \frac{(n+1)!(z^n - 1) z^{n(n-1)/2}}{2(z-1)}.$$

Solution I by Aleksandar Ilić, student, University of Niš, Serbia. Let $S[n, k] = \sum_{P \subseteq [n]} (-1)^{|P|} s^k(z, P)$. We prove the identities by induction on n. It holds by inspection that $S[1, 0] = 0$, $S[1, 1] = -1$, and $S[1, 2] = -1$. For the induction step, place the subsets of $[n+1]$ into two groups: those that contain n and those that do not. Using the binomial theorem,

$$\begin{aligned} S[n+1, k] &= \sum_{P \subseteq [n]} (-1)^{|P|} s^k(z, P) + \sum_{P \subseteq [n]} (-1)^{|P|+1} \left(s(z, P) + z^n\right)^k \\ &= \sum_{P \subseteq [n]} (-1)^{|P|} s^k(z, P) - \sum_{P \subseteq [n]} (-1)^{|P|} \left(\sum_{i=0}^{k} \binom{k}{i} s^i(z, P) z^{n(k-i)} \right). \end{aligned}$$

After canceling equal terms and interchanging sums,

$$S[n+1,k] = -\sum_{i=0}^{k-1} \binom{k}{i} z^{n(k-i)} \sum_{P\subseteq[n]} (-1)^{|P|} s^i(z,P) = -\sum_{i=0}^{k-1} \binom{k}{i} z^{n(k-i)} S[n,i].$$

For $0 \le k \le n$, the induction hypothesis yields $S[n+1,k] = 0$. For $k = n+1$,

$$S[n+1,n+1] = -\binom{n+1}{n} z^n S[n,n] = (-1)^{n+1}(n+1)!\, z^{n(n+1)/2}.$$

For $k = n+2$, we simply extract common factors:

$$\begin{aligned} S[n+1,n+2] &= -\binom{n+2}{n} z^{2n} S[n,n] - \binom{n+2}{n+1} z^n S[n,n+1] \\ &= (-1)^{n+1} \frac{(n+2)!}{2} z^n z^{n(n-1)/2} \left[z^n + \frac{z^n - 1}{z-1} \right] \\ &= (-1)^{n+1} \frac{(n+2)!\,(z^{n+1}-1) z^{n(n+1)/2}}{2(z-1)}. \end{aligned}$$

Solution II by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. We have

$$\begin{aligned} \prod_{m=0}^{n-1} \left(1 - e^{yz^m}\right) &= \sum_{P\subseteq[n]} (-1)^{|P|} e^{ys(z,P)} = \sum_{P\subseteq[n]} (-1)^{|P|} \sum_{k=0}^{\infty} \frac{y^k}{k!} s^k(z,P) \\ &= \sum_{k=0}^{\infty} \frac{y^k}{k!} \sum_{P\subseteq[n]} (-1)^{|P|} s^k(z,P) \end{aligned}$$

and also

$$\begin{aligned} \prod_{m=0}^{n-1} \left(1 - e^{yz^m}\right) &= (-1)^n \prod_{m=0}^{n-1} \left(yz^m + \frac{1}{2} y^2 z^{2m} + \mathcal{O}(y^3) \right) \\ &= (-1)^n y^n z^{n(n-1)/2} + (-1)^n \frac{1}{2} y^{n+1} z^{n(n-1)/2} \sum_{m=0}^{n-1} z^m + \mathcal{O}(y^{n+2}), \end{aligned}$$

where $\mathcal{O}(y^t)$ indicates a series divisible by y^t. Equating powers of y in these two expressions gives the required equalities.

Also solved by U. Abel (Germany), S. Amghibech (Canada), M. R. Avidon, D. Beckwith, K. Bernstein, N. Caro (Brazil), R. Chapman (U. K.), P. Corn, P. P. Dályay (Hungary), S. M. Gagola Jr., J. Grivaux (France), E. A. Herman, J. H. Lindsey II, U. Milutinović (Slovenia), A. Nijenhuis, M. A. Prasad (India), N. C. Singer, R. J. Snelling, A. Stadler (Switzerland), A. Stenger, R. Stong, M. Tetiva (Romania), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

If *AB* Preserves *A*, then *BA* Preserves *B*

11239 [2006, 655]. *Proposed by Michel Bataille, Rouen, France.* Let A and B be complex $n \times n$ matrices of the same rank. Show that if $A^2B = A$, then $B^2A = B$.

Solution by John W. Hagood, Northern Arizona University, Flagstaff, AZ. If $A^2B = A$, then rank $A \ge$ rank $A^2 \ge$ rank A. Thus A, A^2, and B all have the same rank, and hence their null spaces have the same dimension. Since the null spaces of A^2 and B are subspaces of the null space of A, the three null spaces are identical. If $z \in \mathbb{C}^n$, then $A^2B(Az) = A(Az)$, so $A^2(BAz - z) = 0$. This yields $B(BAz - z) = 0$, since the null spaces are identical. Since z is arbitrary, the conclusion follows.

Editorial comment. Jeff Stuart proved the following generalization: if $A^kB = A^{k-1}$ and $\operatorname{rank}(A^{k-1}) = \operatorname{rank}(B^{k-1})$ for some integer k greater than 1, then $B^kA = B^{k-1}$.

Also solved by A. Aguado & G. F. Seelinger, A. Alikhani & A. Dehkordi (Iran), S. Amghibech (Canada), M. Barr (Canada), P. Budney, R. Chapman (U. K.), P. R. Chernoff, K. Dale (Norway), P. P. Dályay (Hungary), L. M. DeAlba, G. Dospinescu (France), M. Goldenberg & M. Kaplan, J. Hartman, E. A. Herman, R. A. Horn, A. K. Shaffie (Iran), G. Keselman, J. H. Lindsey II, O. P. Lossers (Netherlands), S. Rosset, K. Schilling, N. C. Singer, J. H. Smith, A. Stadler (Switzerland), R. Stong, J. Stuart, T. Tam, X. Wang, BSI Problems Group (Germany), Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

Fibonacci Numbers and Tiling a Board with Cuts

11241 [2006, 656]. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata", Rome, Italy.* Find a closed formula for

$$\sum_{k=0}^{n} 2^{n-k} \sum_{x \in S[k,n]} \prod_{i=1}^{k+1} F_{1+2x_i},$$

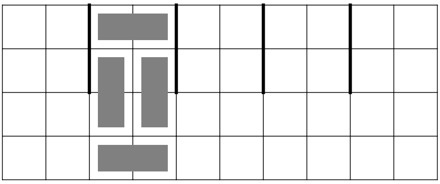
where F_n denotes the nth Fibonacci number (that is, $F_0 = 0$, $F_1 = 1$, and $F_j = F_{j-1} + F_{j-2}$ when $j \geq 2$) and $S[k, n]$ is the set of all $(k+1)$-tuples of nonnegative integers that sum to $n - k$.

Solution I by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. Since $\sum x_i = n - k$ for $x \in S[k, n]$, we can draw the factors of 2 into the terms of the summation and obtain $\sum_{k=0}^{n} \sum_{x \in S[k,n]} \prod_{i=1}^{k+1} G_{x_i}$, where $G_i = 2^i F_{1+2i}$. The Fibonacci recurrence yields $F_{n+2} = 3F_n - F_{n-2}$, so the sequence $\langle G_i \rangle$ is defined by $G_0 = 1$, $G_1 = 4$, and $G_{n+2} = 6G_{n+1} - 4G_n$ for $n \geq 0$. The generating function $G(y)$ of this sequence is given by

$$G(y) = \sum_{i=0}^{\infty} G_i y^i = \frac{1 - 2y}{1 - 6y + 4y^2}.$$

The desired expression is the coefficient of y^n in $\sum_{k=0}^{\infty} y^k G^{k+1}(y)$. Hence its generating function is $\frac{G(y)}{1 - yG(y)}$, which equals $\frac{1-2y}{1-7y+6y^2}$. This expands by partial fractions to $\frac{1}{5}(1 - y)^{-1} + \frac{4}{5}(1 - 6y)^{-1}$. Hence the coefficient of y^n is $(1 + 4 \cdot 6^n)/5$.

Solution II by the proposer. Let a_n be the number of domino tilings of a 4-by-$2n$ region $[0, 2n] \times [0, 4]$ in which dominos are not allowed to cross the $n - 1$ vertical cuts from $(2i, 2)$ to $(2i, 4)$ for $1 \leq i \leq n - 1$. The cuts are indicated by heavy lines in the figure below for $n = 5$.



We show first that the given expression equals a_n. Call each 4-by-2 rectangle a *tooth*. If the top part of a tooth is not tiled with two horizontal dominos or two vertical dominos, then the tooth is tiled like the second tooth shown above. Let k be the number of teeth tiled in this way. For each of the $n-k$ remaining teeth, the top part can be tiled in two ways. This leaves regions that are 2-by-$2x_i$ rectangles, where $x_0, \dots, x_k$ is an element of $S[k,n]$. It is well known (by induction using the Fibonacci recurrence) that the number of domino tilings of a 2-by-m rectangle is F_{1+m}. Hence we have proved that the number of tilings equals the given sum.

Next we obtain a recurrence for the sequence $\langle a_n \rangle$ by considering how the last tooth can be tiled. Consider whether the domino covering the lower right corner is horizontal or vertical. If it is horizontal, then no domino crosses from the last tooth to the one before it, and there are three ways to tile the rest of the last tooth. Hence there are $3a_{n-1}$ tilings of this type.

If the lower right domino is vertical, then there are two ways to tile the top part of the tooth, and what remains is a 4-by-$2(n-1)$ region with the two lower rows extended by one unit. Let b_{n-1} be the number of ways to tile this region. With two ways to tile the upper half of the last tooth, we have $a_n = 3a_{n-1} + 2b_{n-1}$.

For b_{n-1}, we also consider whether the domino covering the lower right corner is horizontal or vertical. If vertical, then there are a_{n-1} ways to complete the tiling. If horizontal, then there is another horizontal tile above it, two ways to tile the top half of tooth $n-1$, and b_{n-2} ways to complete the rest. Hence $b_{n-1} = a_{n-1} + 2b_{n-2}$.

Using the first recurrence to substitute into the second and eliminate the auxiliary sequence yields $a_n = 7a_{n-1} - 6a_{n-2}$. With $a_0 = 1$ and $a_1 = 5$, the solution is $a_n = (1 + 4 \cdot 6^n)/5$.

Also solved by S. Amghibech (Canada), D. Beckwith, R. Chapman (U. K.), P. P. Dályay (Hungary), J. W. Frommeyer, C. C. Heckman, G. Keselman, K. McInturff, N. C. Singer, A. Stadler (Switzerland), A. Stenger, R. Stong, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

"The Ballot Problem in Disguise", in Disguise

11249 [2006, 760]. *Proposed by David Beckwith, Sag Harbor, NY.* A *node-labeled rooted tree* is a tree such that any parent with label ℓ has $\ell+1$ children, labeled $1, 2, \dots, \ell+1$, and such that the root vertex (generation 0) has label 1. Find the population of generation n.

Solution I by Michael R. Avidon, Allston, MA. The answer is $\binom{2n}{n} - \binom{2n}{n+2}$. Let $P(n)$ be the population of generation n and $P_j(n)$ be the number with label j. Thus $P(n) = \sum_{j=1}^{n+1} P_j(n)$. Children with label j come (one each) from parents with label at least $j-1$, so $P_j(n) = \sum_{i=j-1}^{n} P_i(n-1)$. We claim that $P_j(n) = \binom{2n-j}{n-1} - \binom{2n-j}{n+1}$ for $n \geq 1$ and $1 \leq j \leq n+1$. Note that this holds for $n = 1$. Inductively, since $\sum_{i=k}^{m} \binom{i}{k} = \binom{m+1}{k+1}$, summing $\sum_{i=j-1}^{n} P_i(n-1)$ yields

$$P_j(n) = \sum_{i=j-1}^{n} \binom{2n-2-i}{n-2} - \sum_{i=j-1}^{n} \binom{2n-2-i}{n} = \binom{2n-j}{n-1} - \binom{2n-j}{n+1}.$$

Now evaluate $\sum_{j=1}^{n+1} P_j(n)$ to find $P(n)$.

Solution II by Li Zhou, Polk Community College, Winter Haven, FL. The answer is the Catalan number C_{n+1}, which equals $\frac{1}{n+2}\binom{2n+2}{n+1}$. Identify each member of generation n with the list $(a_0, \dots, a_n)$ of labels on the path to it from the root. For $n \geq 0$, let $s_{n+1} = |S_{n+1}|$, where $S_{n+1} = \{(a_0, \dots, a_n)\colon\ a_0 = 1 \text{ and } 1 \le a_i \le a_{i-1} + 1 \text{ for } 1 \le i \le n\}$. We take $s_0 = 1$, since S_0 consists only of the empty 0-tuple. For $n \ge 1$, consider $(a_0, \dots, a_n) \in S_{n+1}$. When $a_1 = 1$, let $k = 0$. When $a_1 \ge 2$, let k be the largest i such that $a_j \ge 2$ for $1 \le j \le i$. Now $(a_1 - 1, \dots, a_k - 1) \in S_k$ and $(a_{k+1}, \dots, a_n) \in S_{n-k}$. Grouping by k yields $s_{n+1} = \sum_{k=0}^{n} s_k s_{n-k}$, which is the well-known Catalan recurrence.

Solution III by Richard Stong, Rice University, Houston, TX. The answer is the Catalan number C_{n+1}, which counts the paths from $(0, 0)$ to $(2n + 2, 0)$ with up-steps $(+1, +1)$ and down-steps $(+1, -1)$ that never go below the x-axis ("Dyck paths"). We exhibit a bijection from such paths to the vertices in generation n. From a path, we form a list $(a_0, \dots, a_n)$ by letting a_k be the height of the path after the $(k + 1)$th up-step. Note that $a_0 = 1$. If $a_k = \ell$, then a_{k+1} can be any of $\{1, \dots, \ell + 1\}$, since between 0 and ℓ down-steps before can occur before the next up-step. Furthermore, each element of the set of path-labels to vertices of generation n arises exactly once under this map.

Editorial comment. Several solvers noted that $P_j(n)$ is the appropriate entry in the Catalan triangle (see sequence A009766 in *The On-Line Encyclopedia of Integer Sequences*). After shifting the index by 1, that value is in fact what is requested in Problem E3402 of this MONTHLY [1990, 612], with essentially the same solution ("The Ballot Problem in Disguise", [1992, 367–368]) as in Solution I here. Robin Chapman mentioned that these trees appear in two papers by Julian West (*Discrete Mathematics* **146** (1995) 247–262 and **157** (1996) 363–374). Daniele Degiorgi pointed out that a solution appears in Donald Knuth's *The Art of Computer Programming*, Volume 4, page 13. The BSI Problems Group observed that this is essentially one of the 66 characterizations of the Catalan numbers given in Exercise 6.19 of Richard Stanley's *Enumerative Combinatorics*, Volume 2.

Also solved by T. Achenbach, M. R. Bacon & C. K. Cook, J. C. Binz (Switzerland), N. Caro (Brazil), R. Chapman (U. K.), A. Chaudhuri, K. Dale (Norway), P. P. Dályay (Hungary), D. Degiorgi (Switzerland), J.-P. Grivaux (France), M. Hudelson, G. Keselman, R. A. Kopas, Y.-J. Kuo, H. Kwong, C. F. Letsche, J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (U. K.), K. McInturff, M. A. Prasad (India), R. E. Prather, V. Schindler (Germany), E. Schmeichel, R. Tauraso (Italy), D. Walsh, BSI Problems Group (Germany), Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

A Permanent Lower Bound

11253 [2006, 847]. *Proposed by David Beckwith, Sag Harbor, NY.* Let n be a positive integer and A be an $n \times n$ matrix with all entries $a_{i,j}$ positive. Let P be the permanent of A. Prove that

$$P \ge n! \prod_{1 \le i,j \le n} a_{i,j}^{1/n}.$$

Solution by Michel Bataille, Rouen, France. Let $\mathcal{S}_n$ be the set of permutations of $\{1, \dots, n\}$. Since $P = \sum_{\sigma \in \mathcal{S}_n} a_\sigma$, where $a_\sigma = \prod_{i=1}^{n} a_{i,\sigma(i)} > 0$, the arithmetic-geometric mean inequality yields $P/n! \ge \left(\prod_{\sigma \in \mathcal{S}_n} a_\sigma\right)^{1/n!}$. Now

$$\prod_{\sigma \in \mathcal{S}_n} a_\sigma = \prod_{i=1}^{n} \left(\prod_{\sigma \in \mathcal{S}_n} a_{i,\sigma(i)} \right) = \prod_{i=1}^{n} \prod_{j=1}^{n} a_{i,j}^{(n-1)!},$$

and the result follows.

Editorial comment. Equality holds if and only if all a_σ are equal, which requires that $a_{ij} = \alpha_i \beta_j$ for all i and j, where all α_i and β_j are positive numbers.

Also solved by S. Amghibech (Canada), J. C. Binz (Switzerland), P. Budney, R. Chapman (U. K.), K. Dale (Norway), P. P. Dályay (Hungary), S. M. Gagola Jr., J.-P. Grivaux (France), E. A. Herman, S. Hitotumatu (Japan), D. Karagulyan (Armenia), P. T. Krasopoulos (Greece), O. López & N. Caro (Brazil), O. P. Lossers (Netherlands), R. Martin (U. K.), J. Rooin & M. A. Maleki (Iran), K. Schilling, V. Schindler (Germany), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), J. Vinuesa (Spain), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.

Primes with Special Primitive Roots

11254 [2006, 847]. *Proposed by Lenny Jones, Shippensburg University, Shippensburg, PA.* For a prime p greater than 3, let S_p be the set of positive integers less than $(p-1)/2$ and relatively prime to $p-1$. Characterize the primes p for which there exists a primitive root g modulo p such that the product of g^a, taken over all a in S_p, is also a primitive root modulo p.

Solution by Toni Ernvall, University of Turku, Finland. Let g be a primitive root modulo p. The product of g^a, taken over all a in S_p, is g^s, where $s = \sum_{a \in S_p} a$, and it is a primitive root modulo p if and only if $\gcd(s, p-1) = 1$. We will prove that this happens only when $p = 5$ and when p has the form $2q^m + 1$, where m is a positive integer, q is a prime, and $q \equiv 3 \pmod 4$.

Suppose first that $p \equiv 1 \pmod 4$. If $p = 5$, then $S_p = \{1\}$ and any primitive root works. For $p \neq 5$, we have $a \in S_p$ if and only if $(p-1)/2 - a \in S_p$. Since $(p-1)/4 \notin S_p$, this makes $|S_p|$ even. Since every element of S_p is odd, s is even and $\gcd(s, p-1) \neq 1$. Hence p is not such a prime.

Next consider $p \equiv 3 \pmod 4$, with $\gcd(s, p-1) = 1$. The set of positive numbers less than $p-1$ and relatively prime to $p-1$ consists of S_p and those a such that $p-1-a \in S_p$. Hence $|S_p| = \frac{1}{2}\phi(p-1)$. Since elements of S_p are odd, $|S_p| \equiv s \pmod 2$. Thus $\gcd(s, p-1) = 1$ forces $|S_p|$ to be odd, and hence $\phi(p-1) \equiv 2 \pmod 4$.

Always $\phi(n) = n \prod_{q \in P(n)} \frac{q-1}{q}$, where $P(n)$ is the set of distinct prime factors of n. When $n \equiv 2 \pmod 4$, the number of factors of 2 dividing $\phi(n)$ is thus the number of distinct odd prime factors of n, plus 1 or more for each that is congruent to 1 modulo 4. Since $\phi(p-1) \equiv 2 \pmod 4$, we conclude that $p - 1 = 2q^m$, where m is a positive integer, q is a prime, and $q \equiv 3 \pmod 4$.

For sufficiency, we compute s for such a prime p. We sum the odd integers less than $(p-1)/2$ that are not divisible by q. Since $q \mid (p-1)/2$, we sum the odd numbers up to $(p-1)/2$ and subtract q times the sum of the odd numbers up to $(p-1)/2q$. Using $p - 1 = 2q^m$, we obtain

$$s = \left(\frac{p+1}{4}\right)^2 - q\frac{1}{4}\left(\frac{p-1}{2q} + 1\right)^2 = \frac{1}{4}(q^m + 1)^2 - \frac{q}{4}(q^{m-1} + 1)^2.$$

After simplifying, s equals $\frac{1}{4}(q-1)(q^{2m-1} - 1)$, which is relatively prime to $2q^m$. Hence g^s is a primitive root, as desired.

Also solved by M. R. Avidon, J. C. Binz (Switzerland), J. Christopher, S. M. Gagola Jr., K. Goldenberg & M. Kaplan, D. E. Ianucci, O. P. Lossers (Netherlands), B. Schmuland (Canada), A. Stadler (Switzerland), V. Stakhovsky, R. Stong, M. Tetiva (Romania), A. Wyn-Jones, Armstrong Problem Solvers, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

Never a Square

11259 [2006, 939]. *Proposed by Nobuhisa Abe, NBU Attached Senior High School, Saiki, Japan.* For integers n greater than 2, let

$$f(n) = \sum_{j=1}^{n-2} 2^j \sum_S \prod_{k \in S} k,$$

where the sum is over all j-element subsets S of the set $\{1, \dots, n-1\}$. Show that $4(2n-1)! + (f(n))^2$ is never the square of an integer.

Solution by Marian Tetiva, Bîrlad, Romania. By expanding $\prod_{k=1}^{n-1}(1 + c_k x)$ and setting $c_k = k$ and $x = 2$, we obtain $f(n) = \prod_{k=1}^{n-1}(2k+1) - (n-1)!2^{n-1} - 1$. Letting $a = \prod_{k=1}^{n-1}(2k+1)$ and $b = \prod_{k=1}^{n-1} 2k$, we have $f(n) = a - b - 1$, and our task is to show that $4ab + (a-b-1)^2$ is not a square. This follows from

$$(a+b-1)^2 < 4ab + (a-b-1)^2 < (a+b)^2,$$

which holds whenever $a > b > 0$. That condition holds when $n \geq 2$.

Also solved by S. Amghibech (Canada), A. Bandeira & J. Moreira (Portugal), D. Beckwith, R. Chapman (U. K.), P. P. Dályay (Hungary), O. Kouba (Syria), O. P. Lossers (Netherlands), C. R. Pranesachar (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), BSI Problems Group (Germany), FAU Problem Solving Group, and the proposer.

Getting to Know You, Directly and Indirectly

11262 [2006, 940]. *Proposed by Ashay Burungale, Satara, Maharashtra, India.* In a certain town of population $2n+1$, one knows those to whom one is known. For any set A of n citizens, there is some person among the other $n+1$ who knows everyone in A. Show that some citizen of the town knows all the others.

Solution I by Kenneth Schilling, University of Michigan—Flint, Flint, MI. A *clique* is a set of citizens who all know one another. If a clique B has fewer than $n+1$ citizens, then by hypothesis someone not in B knows everyone in B, so we may form a larger clique by adding that person to B. Hence there is a clique of size $n+1$. The set A of citizens not in B has size n, so by hypothesis some person in B knows every member of A and hence knows everyone.

Solution II by Christopher Carl Heckman, Arizona State University, Tempe, AZ. We prove the contrapositive: If nobody knows everyone, then some set A of n citizens is not completely known by anyone outside A. We first color each citizen red or green as follows. If p is not yet colored, choose some p' unknown to p. If p' is already colored, give p the opposite color. Otherwise, give p and p' opposite colors. At the end, everyone has a color opposite from that of someone he or she does not know. The less-frequent color occurs at most n times; let A be a set of size n including all those with that color. Each person outside A has the other color and hence does not know everyone in A.

Editorial comment. The problem of course is a statement about graphs (perhaps the popularity of the problem stemmed from not using that language). In the language of graph theory, it states that a graph of odd order has a vertex adjacent to all others if each set with fewer than half the vertices has a common neighbor. Several solvers did use graph theoretic language and arguments.

Also solved by 51 other readers and the proposer.