# Generalizing a Curious Combinatorial Identity

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In [5] Simons proved a binomial coefficient identity using repeated differentiation which can be equivalently written as

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}(1+x)^k=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k.$$

Proofs of this identity have been given by Chapman [1] using generating functions and by Prodinger [3] using Cauchy's integral formula. Using Prodinger's approach Munarini [2] established some generalizations of Simons' identity and other neat identities, some of them proved by Shattuck recently in [4], using elegant combinatorial arguments. In the following we present an alternate generalization, different from the ones given by Munarini. The proof we give is very elementary, based on a simple algebraic manipulation.

**Proposition**. Let $x$, $y$ be arbitrary real numbers, and let $p$, $q$ be two positive integers. Then the following identity holds

$$\sum_{k=0}^{p}\binom{p}{k}\binom{q+k}{k}x^{p-k}y^k=\sum_{k=0}^{p}\binom{p}{k}\binom{q}{k}(x+y)^{p-k}y^k.$$

*Proof*. Let $m$, $n$ be two arbitrary real numbers. Consider the identity:

$$\left[(x+y)m+y\right]^p(1+m)^q=\left(\sum_{k=0}^{p}\binom{p}{k}(x+y)^{p-k}\,m^{p-k}y^k\right)\left(\sum_{k=0}^{q}\binom{q}{k}m^k\right).$$

The coefficient of $m^p$ from the right hand side of the identity is

$$\sum_{k=0}^{p}\binom{p}{k}\binom{q}{k}(x+y)^{p-k}y^k.$$

On other hand, since

$$\left[(x+y)m+y\right]^p(1+m)^q=\left[mx+y(1+m)\right]^p(1+m)^q=\sum_{k=0}^{p}\binom{p}{k}x^{p-k}m^{p-k}y^k(1+m)^{q+k},$$

the coefficient of $m^p$ is also equal to

$$\sum_{k=0}^{p}\binom{p}{k}\binom{q+k}{k}x^{p-k}y^k,$$

and therefore

$$\sum_{k=0}^{p}\binom{p}{k}\binom{q+k}{k}x^{p-k}y^{k}=\sum_{k=0}^{p}\binom{p}{k}\binom{q}{k}(x+y)^{p-k}y^{k}.$$

□

For $p=q=n$, $x=1$, $y=x$, the identity is equivalent with

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^{k}=\sum_{k=0}^{n}\binom{n}{k}^{2}(1+x)^{n-k}x^{k}.$$

For $p=q=n$, $x=-1$, $y=1+x$, the identity is equivalent with

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{p-k}(1+x)^{k}=\sum_{k=0}^{n}\binom{n}{k}^{2}x^{n-k}(1+x)^{k}=\sum_{k=0}^{n}\binom{n}{k}^{2}(1+x)^{n-k}x^{k}.$$

Hence, we deduce Simons' identity

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}(1+x)^{k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^{k}.$$

## References

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4. M. Shattuck. Combinatorial proofs of some Simons-type binomial coefficient identities. *Integers*; 7 (2007), #A27.

5. S. Simons, A curious identity. *The Mathematical Gazette*; 85 (2001), 296298.