An Elementary Proof of Blundon's Inequality

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Abstract

In this note, we give an elementary proof of Blundon's Inequality. We make use of a simple auxiliary result, provable by only using the Arithmetic Mean - Geometric Mean Inequality.

For a given triangle ABC we shall consider that A, B, C denote the magnitudes of its angles, and a, b, c denote the lengths of its corresponding sides. Let R, r and s be the circumradius, the inradius and the semi-perimeter of the triangle, respectively. In addition, we will occasionally make use of the symbols $\sum$ (cyclic sum) and $\prod$ (cyclic product), such as

$$\sum f(a) = f(a) + f(b) + f(c), \quad \prod f(a) = f(a)f(b)f(c).$$

In the AMERICAN MATHEMATICAL MONTHLY, W. J. Blundon [1] asked for the proof of the inequality

$$s \leq 2R + (3\sqrt{3} - 4)r$$

which holds in any triangle ABC. The solution given by the editors is in fact just a comment made by A. Makowski [3], who sends the reader to [2], where Blundon originally published this inequality, and where he actually proves more, namely that this is the best such inequality in the following sense: if, for the numbers k and h the inequality

$$s \leq kR + hr$$

is valid in any triangle, with the equality occurring when the triangle is equilateral, then

$$2R + (3\sqrt{3} - 4)r \leq kR + hr.$$

In this note we give a new proof of Blundon's inequality by making use of the following preliminary result:

Lemma 1. *Any positive real numbers* $x,\ y,\ z$ *such that*

$$x + y + z = xyz$$

satisfy the inequality

$$(x-1)(y-1)(z-1) \leq 6\sqrt{3} - 10.$$

Proof. Since the numbers are positive, from the given condition it follows immediately that $x < xyz \Leftrightarrow yz > 1$, and similarly $xz > 1$ and $yz > 1$, which shows that it is not possible for two of the numbers to be less than or equal to 1 (neither can all the numbers be less than 1). Because if a number is less than 1 and two are greater than 1 the inequality is obviously true (the product from the left-hand side being negative), we still have to consider the case when $x > 1, y > 1, z > 1$. Then the numbers $u = x - 1, v = y - 1$ and $w = z - 1$ are positive and, replacing $x = u + 1, y = v + 1, z = w + 1$ in the condition from the hypothesis, one gets

$$uvw + uv + uw + vw = 2.$$

By the Arithmetic Mean - Geometric Mean inequality

$$uvw + 3\sqrt[3]{u^2v^2w^2} \le uvw + uv + uw + vw = 2,$$

and hence for $t = \sqrt[3]{uvw}$ we have

$$t^3 + 3t^2 - 2 \le 0 \Leftrightarrow (t+1)(t+1+\sqrt{3})(t+1-\sqrt{3}) \le 0.$$

We conclude that $t \le \sqrt{3} - 1$ and thus,

$$(x-1)(y-1)(z-1) \le 6\sqrt{3} - 10.$$

The equality occurs when $x = y = z = \sqrt{3}$. This proves Lemma 1. □

We now proceed to prove Blundon's Inequality.

Theorem 2. *In any triangle ABC, we have that*

$$s \le 2R + (3\sqrt{3} - 4)r.$$

The equality occurs if and only if ABC is equilateral.

Proof. According to the well-known formulae

$$\cot\frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}, \quad \cot\frac{B}{2} = \sqrt{\frac{s(s-b)}{(s-c)(s-a)}}, \quad \cot\frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}},$$

we deduce that

$$\sum \cot\frac{A}{2} = \prod \cot\frac{A}{2} = \frac{s}{r},$$

and

$$\sum \cot\frac{A}{2}\cot\frac{B}{2} = \sum \frac{s}{s-a} = \frac{4R+r}{r}.$$

In this case, by applying Lemma 1 to the positive numbers $x = \cot\frac{A}{2}$, $y = \cot\frac{B}{2}$ and $z = \cot\frac{C}{2}$, it follows that

$$(\cot\frac{A}{2} - 1)(\cot\frac{B}{2} - 1)(\cot\frac{C}{2} - 1) \le 6\sqrt{3} - 10,$$

and therefore

$$2\prod\cot\frac{A}{2}-\left(\sum\cot\frac{A}{2}\cot\frac{B}{2}\right)\leq 6\sqrt{3}-9.$$

This rewrites as

$$\frac{2s}{r}-\frac{4R+r}{r}\leq 6\sqrt{3}-9,$$

and thus

$$s\leq 2R+(3\sqrt{3}-4)r.$$

The equality occurs if and only if $\cot\frac{A}{2}=\cot\frac{B}{2}=\cot\frac{C}{2}$, i.e. when the triangle ABC is equilateral. This completes the proof of Blundon's Inequality. ☐

References

[1] W. J. BLUNDON, Problem E1935, *The Amer. Math. Monthly*, **73**(1966), 1122.

[2] W. J. BLUNDON, Inequalities associated with the triangle, *Canad. Math. Bull.*, **8**(1965), 615-626.

[3] A. MAKOWSKI, Solution of the Problem E1935, *The Amer. Math. Monthly*, **75**(1968), 404.

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