A Combinatorial Proof of Lindström's Theorem on Unions of Subsets of a Finite Set

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Abstract

In this note, we give a combinatorial proof of Bernt Lindström's theorem that if A_1, A_2, …, A_{n+1} are nonempty subsets of an n-element set, then we can find two disjoint and nonempty groups of indices $\{i_1,\ i_2,\ \ldots,\ i_k\}$ and $\{j_1,\ j_2,\ \ldots,\ j_m\}$ such that

$$A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k} = A_{j_1} \cup A_{j_2} \cup \ldots \cup A_{j_m}.$$

1. Introduction

In 1968, B. Landström proposed the beautiful result from above as a problem in the AMERICAN MATHEMATICAL MONTHLY, which was sequently "solved" by H. S. Hahn in [1]. One year after the publication of this proof several readers of the MONTHLY pointed out that Hahn's approach was incorrect (this explains why previously, we used aphostrophes around the word *solved*) and furthermore, E. Smet gave a *correct* solution by using notions of linear algebra. For the sake of completeness, we shall record this proof below.

Lindström's Theorem. Consider a set S of n elements and $n+1$ subsets $A_1, A_2, \ldots, A_n \subseteq S$. Then, there exist $k,\ m \geq 1$ and two disjoint and nonempty sets of indices $\{i_1,\ i_2,\ \ldots,\ i_k\} \cap \{j_1,\ j_2,\ \ldots,\ j_m\} = \emptyset$ such that

$$A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k} = A_{j_1} \cup A_{j_2} \cup \ldots \cup A_{j_m}.$$

Proof (due to `Smet` [2]). Without loss of generality, we may assume that $S = \{a_1, a_2, \ldots, a_n\}$. We associate to the subset A_j the vector $v_j \in \mathbb{R}^n$ having the components $v_j = (v_{1j}, v_{2j}, \ldots, v_{nj})$, given by

$$v_{ij} = \begin{cases} 1, \text{if } a_i \in A_j; \\ 0, \text{if } a_i \notin A_j. \end{cases}$$

Since we have $n+1$ vectors $v_1,\ v_2,\ \ldots,\ v_{n+1}$ in $\mathbb{R}^n$, there exists a nontrivial linear combination of them that vanishes. We separate the positive coefficients and the negative coefficients and write down this combination in the form

$$\sum_{i \in I} \lambda_i v_i = \sum_{j \in J} \lambda_j v_j,$$

where $I,\ J$ are disjoint sets of indices and $\lambda_h > 0$, for $h \in I \cup J$. We claim now that

$$\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j.$$

In fact, assume that $a_h \in \bigcup_{i \in I} A_i$ and thus $a_h \in A_i$ for some $i \in I$. Then the h-th coordinate of the vector v_i is nonzero. All coordinates of the vectors v_i are nonnegative and the coefficients λ_i

are positive, which implies that the h-th coordinate of the vector $\sum_{i\in I}\lambda_i v_i$ is nonzero. Now, since this h-th coordinate of $\sum_{j\in J}\lambda_j v_j$ is nonzero, there should exist some $j\in J$ for which the h-th coordinate of v_j is nonzero and hence $a_i\in A_j\subset\bigcup_{j\in J}A_j$. This proves that $\bigcup_{i\in I}A_i\subset\bigcup_{j\in J}A_j$, and by symmetry we can analogously establish that $\bigcup_{j\in J}A_j\subset\bigcup_{i\in I}A_i$. This yields

$$A_{i_1}\cup A_{i_2}\cup\ldots\cup A_{i_k}=A_{j_1}\cup A_{j_2}\cup\ldots\cup A_{j_m}.$$

□

2. Combinatorial Proof of Lindström's Theorem

We now give a different approach, by making use of induction on n. To ease our work (basically to get rid of some indices), we shall now assume without loss of generality that S is the set of the first n positive integers. Indeed, when $n=1$, we have $A_1=A_2=\{1\}$, the conclusion becoming trivially true.

We proceed to the next step: under the assumption that the statement we have to show holds for n, we will prove it for $n+1$. In this case, A_1, A_2, …, A_{n+2} are the $n+2$ nonempty subsets of $\{1,\ 2,\ \ldots,\ n+1\}$, for which consider $B_i=A_i-\{n+1\}$. We shall further split the problem into three separate cases:

`Case I`. There exist $1\le i<j\le n+2$ such that $B_i=B_j=\emptyset$. Then $A_i=A_j=\{n+1\}$ and we are done.

`Case II`. There exists only one i such that $B_i=\emptyset$. Note that there is no loss of generality in supposing $B_{n+2}=\emptyset$, and so $A_{n+2}=\{n+1\}$. According to the inductive assumption, for $\{1,\ 2,\ \ldots,\ n+1\}$, there exist two disjoint subsets $\{i_1,\ i_2,\ \ldots,\ i_k\}$ and $\{j_1,\ j_2,\ \ldots,\ j_m\}$ such that

$$B_{i_1}\cup B_{i_2}\cup\ldots\cup B_{i_k}=B_{j_1}\cup B_{j_2}\cup\ldots\cup B_{j_m}.$$

Setting $C=A_{i_1}\cup A_{i_2}\cup\ldots\cup A_{i_k}$, $D=A_{j_1}\cup A_{j_2}\cup\ldots\cup A_{j_m}$, we now see that C and D differ at the most by the element $n+1$ (this follows immediately from the definition of B_i). In this case, by putting A_{n+2} into C or D, we get to our conclusion, and so we are done.

`Case III`. No B_i is empty. In this case, since B_1, B_2, …, B_{n+1} are all nonempty subsets of $\{1,\ 2,\ \ldots,\ n\}$, and by the inductive assumption, there exist two disjoint subsets $\{i_1,\ i_2,\ \ldots,\ i_k\}$ and $\{j_1,\ j_2,\ \ldots,\ j_m\}$ such that

$$B_{i_1}\cup B_{i_2}\cup\ldots\cup B_{i_k}=B_{j_1}\cup B_{j_2}\cup\ldots\cup B_{j_m}.$$

In addition, B_2, B_3, …, B_{n+1} are also nonempty subsets of $\{1,\ 2,\ \ldots,\ n\}$, and therefore, according to the inductive assumtpion, we can show that, for $\{2,\ 3,\ \ldots,\ n+2\}$, there exists two disjoint subsets $\{r_1,\ r_2,\ \ldots,\ r_u\}$ and $\{t_1,\ t_2,\ \ldots,\ t_v\}$ such that

$$B_{r_1}\cup B_{r_2}\cup\ldots\cup B_{r_u}=B_{t_1}\cup B_{t_2}\cup\ldots B_{t_v}.$$

Again, we set $C=A_{i_1}\cup A_{i_2}\cup\ldots\cup A_{i_k}$, $D=A_{j_1}\cup A_{j_2}\cup\ldots\cup A_{j_m}$, and furthermore $E=A_{r_1}\cup A_{r_2}\cup\ldots\cup A_{r_u}$, $F=A_{t_1}\cup A_{t_2}\cup\ldots\cup A_{t_v}$. By using the definition of B_i in combination with

$$B_{i_1}\cup B_{i_2}\cup\ldots\cup B_{i_k}=B_{j_1}\cup B_{j_2}\cup\ldots\cup B_{j_m}$$

and

$$B_{r_1} \cup B_{r_2} \cup \ldots \cup B_{r_u} = B_{t_1} \cup B_{t_2} \cup \ldots B_{t_v},$$

we see that C and D differ at the most by the element $n+1$, and so do E and F. If $C = D$ or $E = F$, we are done. Hence it suffices to consider the case when $C \neq D$ and $E \neq F$. Moreover, there is no loss of generality in supposing $C = D \cup \{n+1\}$, which instantly implies $E = F - \{n+1\}$. Now note that $C \cup E = D \cup F$. After amalgamating the sets occurred repeatedly in C and E, as well as in D and F, we get two subsets $\{p_1,\ p_2,\ \ldots,\ p_x\}$ and $\{q_1,\ q_2\ \ldots,\ q_y\}$ of $\{1,\ 2,\ \ldots, n+2\}$ such that

$$A_{p_1} \cup A_{p_2} \cup \ldots \cup A_{p_x} = A_{q_1} \cup A_{q_2} \cup \ldots \cup A_{q_y},$$

where $G = A_{p_1} \cup A_{p_2} \cup \ldots \cup A_{p_x} = C \cup E$, $H = A_{q_1} \cup A_{q_2} \cup \ldots \cup A_{q_y} = D \cup F$.

Now, if $\{p_1,\ p_2,\ \ldots,\ p_x\} \cap \{q_1,\ q_2\ \ldots,\ q_y\} = \emptyset$, we are done. Otherwise, if there exists $i \in \{p_1,\ p_2,\ \ldots,\ p_x\} \cap \{q_1,\ q_2\ \ldots,\ q_y\}$, we write $\tilde{C} = \{A_{i_1},\ A_{i_2},\ \ldots,\ A_{i_k}\}$, $\tilde{D} = \{A_{j_1},\ A_{j_2},\ \ldots,\ A_{j_m}\}$, $\tilde{E} = \{A_{r_1},\ A_{r_2},\ \ldots,\ A_{r_u}\}$, $\tilde{F} = \{A_{t_1},\ A_{t_2},\ \ldots,\ A_{t_v}\}$, and there is no loss of generality in assuming that A_i does not belong to $\tilde{C}$ and $\tilde{E}$ at the same time, and it does not belong to $\tilde{D}$ and $\tilde{F}$ at the same time too. Hence there are only to possibilities:

(a) $A_i \in \tilde{C}$ and $A_i \in \tilde{F}$. If there are two sets in $\tilde{C}$ containing $n+1$, then we take away set A_i from the left side in

$$A_{p_1} \cup A_{p_2} \cup \ldots \cup A_{p_x} = A_{q_1} \cup A_{q_2} \cup \ldots \cup A_{q_y}.$$

Now since all elements except $n+1$ in A_i belong to E, and there are two sets on the left side in the above equation contaning $n+1$. Thus after taking away A_i, the number of elements in G does not reduce and the relation we have just written above remains an equality. In the same way, if there are two sets in $\tilde{F}$ containing $n+1$, then we take away A_i from the same right side, and we still have equality.

Of course, if there is only one set in $\tilde{C}$ and $\tilde{F}$ containing $n+1$, then after taking away A_i from both sides, it remains to be an equality. (Now, by

$$B_{i_1} \cup B_{i_2} \cup \ldots \cup B_{i_k} = B_{j_1} \cup B_{j_2} \cup \ldots \cup B_{j_m} \text{ and } B_{i_1} \cup B_{i_2} \cup \ldots \cup B_{i_k} = B_{j_1} \cup B_{j_2} \cup \ldots \cup B_{j_m},$$

we can see that the two sides of

$$A_{p_1} \cup A_{p_2} \cup \ldots \cup A_{p_x} = A_{q_1} \cup A_{q_2} \cup \ldots \cup A_{q_y}$$

will not become empty sets.)

(b) $A_i \in \tilde{D}$ and $A_i \in \tilde{E}$, then $n+1 \notin A_i$. Now after taking away A_i from both sides of the same

$$A_{p_1} \cup A_{p_2} \cup \ldots \cup A_{p_x} = A_{q_1} \cup A_{q_2} \cup \ldots \cup A_{q_y},$$

the resulting expression is still an equality.

In view of the above operation, we have a method to make the two sets of subscrips $\{p_1,\ p_2,\ \ldots,\ p_x\}$ and $\{q_1,\ q_2\ \ldots,\ q_y\}$ to be disjoint. Therefore we conclude that the statement in question holds for $n+1$. This completes the proof of Lindström's theorem.

References

[1] B. Lindström, H. S. Hahn, Solution to Problem E2106, this MONTHLY, **76** (1969) 697.

[2] B. Lindström, E. Smet, Solution to Problem E2106, this MONTHLY, **77** (1970) 193.

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