On a Theorem Regarding Lattice Pentagons

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Abstract

In this note, we give a short proof of a Russian refinement of Arkinstall's lattice pentagon theorem.

In this paper we will do ourselves a favour and name by *lattice points* just the points from the Euclidean plane $\mathbb{R}^2$ beloning to $\mathbb{Z} \times \mathbb{Z}$. In addition, we will call a pentagon having its vertices as lattice points as a *lattice pentagon*.

In 1980, Arkinstall [2] proved the following result, known nowadays as *the lattice pentagon theorem*:

THEOREM 1. *A convex lattice pentagon must contain a lattice point in its interior.*

Twenty years later, a refinement of this result appeared as a proposed problem in the Russian Mathematical Olympiad.

THEOREM 2. *If $\mathcal{P}$ is a convex pentagon having the vertices on lattice points, the smaller pentagon $S\left(\mathcal{P}\right)$ determined by its diagonals contains, in its interior or on its boundary, a lattice point.*

Several approaches using Pick's celebrated formula (see, for example, [3]) lead to proofs of this beautiful theorem. The most frequent one in this spirit is the one from [1], which combines Pick's theorem with an extremal argument. In the following we give a proof which avoids Pick's formula, based on Arkinstall's original idea from [2].

Proof of Theorem 2. Let A, B, C, D, E be the vertices of $\mathcal{P}$. Since a lattice point can be one of the four types: (2$\mathbb{Z}$+1, 2$\mathbb{Z}$+1), (2$\mathbb{Z}$, 2$\mathbb{Z}$), (2$\mathbb{Z}$+1, 2$\mathbb{Z}$), (2$\mathbb{Z}$, 2$\mathbb{Z}$+1), according to the pidgenhole principle, two of the vertices of $ABCDE$ will have the same form, and therefore the midpoint M of the segment determined by these two points is also a lattice point. We now distinguish two cases:

1. The point M lies in the interior of a diagonal of the pentagon $ABCDE$.

Without loss of generality, assume that M is the midpoint of segment AC. If it lies on the boundary of $S(ABCDE)$ we are done. If it doesn't, one of the two pentagons $ABMDE$ and $MBCDE$ is convex, say $MBCDE$. This convex pentagon is a lattice pentagon contained in the initial $ABCDE$. We observe that $S(MBCDE)$ is included in $S(ABCDE)$, as well. Thus, by inducting on the number of lattice points contained in the initial pentagon $\mathcal{P}$, $S(MBCDE)$ will contain a lattice point, and then so will $S(ABCDE)$, as desired.

2. The point M lies in the interior of an edge of the pentagon $ABCDE$.

Without loss of generality, assume that M is the midpoint of AB. The pentagon $MBCDE$ is convex. Although $S(MBCDE)$ is not included anymore in $S(ABCDE)$, we see that $S(MBCDE)$ is included in the reunion of $S(ABCDE)$ with the triangle **T** bounded by the three lines BD, BE, AC. According to a similar inductive argument, $S(MBCDE)$ contains a lattice point M'. If M'

lies in the interior of $S(ABCDE)$ we are done. If it doesn't, it lies inside triangle **T**, and since the pentagon $AM'CDE$ is convex, with its small pentagon $S(AM'CDE)$ included in $S(ABCDE)$, by applying the same inductive argument, we conclude that $S(AM'CDE)$ contains a lattice point, which lies as well inside $S(ABCDE)$. This proves Theorem 2.

References

[1] T. Andreescu, Z. Feng, G. Lee, Mathematical Olympiads, 2000-2001: Problems and Solutions from Around the World, *MAA*, 2003.

[2] J. R. Arkinstall, Minimal Requirements for Minkowskis Theorem in the Plane I, *Bulletin of the Australian Mathematical Society.*, 22(1980)259274.

[3] G. Pick, Geometrisches zur Zahlentheorie, *Sitzenber. Lotos* (Prague) 19, 311-319, 1899.

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