# From Neuberg-Pedoe Back to Hadwiger-Finsler

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**Abstract**

In this note, we give a new parametrized version of the Hadwiger-Finsler Inequality. Within our proof, we make use of a preliminary result, which can be interpreted as a corollary of the Neuberg-Pedoe Inequality.

## 1. Introduction

The Neuberg-Pedoe Inequality is known in literature as a generalization of the following:

**Theorem 1**. *In any triangle $ABC$ with the side lenghts $a, b, c$ and $S$ its area,*

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3}.$$

This inequality is due to Weitzenbock (see [13]), but has also appeared at International Mathematical Olympiad in 1961. In [6], one can find eleven proofs. In fact, in any triangle $ABC$ the following sequence of inequalities is valid:

$$a^2 + b^2 + c^2 \geq ab + bc + ca \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \geq 3\sqrt[3]{a^2b^2c^2} \geq 4S\sqrt{3}.$$

In 1937, Finsler and Hadwiger found a stronger version [7]:

**Theorem 2**. In any triangle $ABC$ with side lengths $a$, $b$, $c$, and area $S$, the following inequality holds:

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3} + (a-b)^2 + (b-c)^2 + (c-a)^2.$$

Five years later, in addition to the refinement gave by Finsler and Hadwiger, Pedoe [9] proved a magnificient generalization of the same Weitzenbock Inequality. In Mitrinovic, Pecaric, and Volenecs' classic *Recent Advances in Geometric Inequalities*, this generalization is refered to as the Neuberg-Pedoe Inequality. See also [8, 10-12].

**Theorem 3**. *Let $a$, $b$, $c$, and $x$, $y$, $z$ be the side lengths of two given triangles $ABC$, $XYZ$ with areas $S$, and $T$, respectively. Then,*

$$a^2\left(y^2 + z^2 - x^2\right) + b^2\left(z^2 + x^2 - y^2\right) + c^2\left(x^2 + y^2 - z^2\right) \geq 16ST,$$

*with equality if and only if the triangles $ABC$ and $XYZ$ are similar.*

Several proofs are available. For example, see [1-5].

In this note, from a corollary of the Neuberg-Pedoe Inequality we go back to the Hadwiger-Finsler Inequality and give a new parametrized version of their refinement. Before we proceed, we record an inoffensive lemma, known to the author as the Conway substitution theorem.

**Theorem 4**. *Let $u$, $v$, $w$ be three reals such that the numbers $v+w$, $w+u$, $u+v$ and $vw+wu+uv$ are all nonnegative. Then, there exists a triangle $XYZ$ with sidelengths $x=YZ=\sqrt{v+w}$, $y=ZX=\sqrt{w+u}$, $z=XY=\sqrt{u+v}$. This triangle satisfies $y^2+z^2-x^2=2u$, $z^2+x^2-y^2=2v$, $x^2+y^2-z^2=2w$. The area $T$ of this triangle equals $T=\frac{1}{2}\sqrt{vw+wu+uv}$. If $X=\angle ZXY, Y=\angle XYZ, Z=\angle YZX$ are the angles of this triangle, then $\cot X=\frac{u}{2T}$, $\cot Y=\frac{v}{2T}$ and $\cot Z=\frac{w}{2T}$.*

*Proof of Theorem 4*. Since the numbers $v+w$, $w+u$, $u+v$ are nonnegative, their square roots $\sqrt{v+w}$, $\sqrt{w+u}$, $\sqrt{u+v}$ exist, and, of course, are nonnegative as well. Now, we have $\sqrt{w+u}+\sqrt{u+v}\geq\sqrt{v+w}$, since

$$\begin{aligned}\sqrt{w+u}+\sqrt{u+v}\geq\sqrt{v+w} \quad &\Longleftarrow \quad (w+u)+(u+v)+2\sqrt{(w+u)(u+v)}\geq v+w\\ &\Longleftarrow \quad 2\sqrt{(w+u)(u+v)}\geq -2u\\ &\Longleftarrow \quad (w+u)(u+v)\geq(-u)^2\\ &\Longleftarrow \quad vw+wu+uv\geq 0.\end{aligned}$$

Similarly, $\sqrt{u+v}+\sqrt{v+w}\geq\sqrt{w+u}$ and $\sqrt{v+w}+\sqrt{w+u}\geq\sqrt{u+v}$. Thus, there exists a triangle XYZ with sidelengths $x=YZ=\sqrt{v+w}$, $y=ZX=\sqrt{w+u}$, $z=XY=\sqrt{u+v}$. We have

$$y^2+z^2-x^2=\left(\sqrt{w+u}\right)^2+\left(\sqrt{u+v}\right)^2-\left(\sqrt{v+w}\right)^2=(w+u)+(u+v)-(v+w)=2u,$$

and similarly $z^2+x^2-y^2=2v$ and $x^2+y^2-z^2=2w$. According now to the fact that

$$\cot Z=\frac{x^2+y^2-z^2}{4T},$$

we deduce that so that $\cot Z=w/2T$, and similarly $\cot X=u/2T$ and $\cot Y=v/2T$.

Since $\cot Y\cdot\cot Z+\cot Z\cdot\cot X+\cot X\cdot\cot Y=1$, what, using these relations, becomes $\frac{v}{2T}\cdot\frac{w}{2T}+\frac{w}{2T}\cdot\frac{u}{2T}+\frac{u}{2T}\cdot\frac{v}{2T}=1$, it follows that $vw+wu+uv=4T^2$, and thus $T=\frac{1}{2}\sqrt{4T^2}=\frac{1}{2}\sqrt{vw+wu+uv}$. This completes the proof of the Conway substitution theorem. ■

## 2. Main result

**Theorem 5**. *Let $ABC$ be a triangle with side lengths $a$, $b$, $c$, and area $S$ and let $x$, $y$, $z$ be three positive real numbers. Then,*

$$a^2+b^2+c^2\geq 4\sqrt{3}S+\frac{2}{x+y+z}\left(\frac{x^2-yz}{x}\cdot a^2+\frac{y^2-zx}{y}\cdot b^2+\frac{z^2-xy}{z}\cdot c^2\right).$$

*Proof of Theorem 5.* We begin with the promised corollary of the Neuberg-Pedoe Inequality:

**Lemma 6**. Let $ABC$ be a triangle with side lengths $a$, $b$, $c$, and area $S$, and let $u$, $v$, $w$ be three reals such that the numbers $v+w$, $w+u$, $u+v$ and $vw+wu+uv$ are all nonnegative. Then,

$$ua^2+vb^2+wc^2 \geq 4\sqrt{vw+wu+uv}\cdot S.$$

*Proof of Lemma 6.* According to Theorem 3, we can construct a triangle with side-lenghts $x=\sqrt{v+w}$, $y=\sqrt{w+u}$, $z=\sqrt{u+v}$ and area $T=\sqrt{vw+wu+uv}/2$. Let this triangle be $XYZ$. In this case, by the Neuberg-Pedoe Inequality, applied for the triangles $ABC$ and $XYZ$, we get that

$$a^2\left(y^2+z^2-x^2\right)+b^2\left(z^2+x^2-y^2\right)+c^2\left(x^2+y^2-z^2\right) \geq 16ST.$$

By the formulas given in the Conway substitution theorem, this becomes equivalent with

$$a^2\cdot 2u+b^2\cdot 2v+c^2\cdot 2w \geq 16S\cdot\frac{1}{2}\sqrt{vw+wu+uv};$$

which simplifies to $ua^2+vb^2+wc^2 \geq 4\sqrt{vw+wu+uv}\cdot S$. This proves Lemma 6. □

Returning to our main theorem, let $m=xyz(x+y+z)-2yz(x^2-yz)$, $n=xyz(x+y+z)-2zx(y^2-zx)$, and $p=xyz(x+y+z)-2xy(z^2-xy)$. The three terms $n+p$, $p+m$, and $m+n$ are all positive, and since

$$mn+np+pm=3x^2y^2z^2(x+y+z)^2 \geq 0,$$

by Lemma 6, we get that

$$\sum_{cyc}\left[xyz(x+y+z)-2yz(x^2-yz)\right]a^2 \geq 4xyz(x+y+z)\sqrt{3}S.$$

This rewrites as

$$\sum_{cyc}\left[(x+y+z)-2\cdot\frac{x^2-yz}{x}\right]a^2 \geq 4(x+y+z)\sqrt{3}S,$$

and, thus,

$$a^2+b^2+c^2 \geq 4\sqrt{3}S+\frac{2}{x+y+z}\left(\frac{x^2-yz}{x}\cdot a^2+\frac{y^2-zx}{y}\cdot b^2+\frac{z^2-xy}{z}\cdot c^2\right).$$

This completes the proof of Theorem 5. ■

Obviously, for $x=a$, $y=b$, $z=c$, and following the fact that

$$a^3+b^3+c^3-3abc=\frac{1}{2}\left(a+b+c\right)\left[(a-b)^2+(b-c)^2+(c-a)^2\right],$$

Theorem 4 becomes equivalent with the Hadwiger-Finsler Inequality.

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